Complex Analysis: Resit Exam

Aletta Jacobshal 01, Friday 12 April 2019, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** on the envelope and at the top of each answer sheet.
- Use the ruled paper for writing the answers and use the blank paper as scratch paper. After finishing put your answers in the envelope. **Do NOT seal the envelope!** You must return the scratch paper and the printed exam (separately from the envelope). The exam and its solutions will be uploaded to Nestor in the following days.
- Solutions should be complete and clearly present your reasoning. **When you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.**
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.

Question 1 (10 points)

Consider a function $f(z)$ such that $\mathrm{Re}(f(z)) \geq M$ for all $z \in \mathbb{C}$, where M is a real constant. Prove that if $f(z)$ is entire then it must be constant. Hint: consider the function $e^{-f(z)}$.

Question 2 (20 points)

(a) (8 points) Consider the integral

$$\mathrm{pv} \int_{-\infty}^{\infty} \frac{e^{4\mathrm{i}x}}{x^2 - 1} \,\mathrm{d}x.$$

Specify and draw a (closed) contour that you can use to compute such an integral with the calculus of residues. Give full justification for your choice of contour. NB: You are *not* being asked to compute this integral.

(b) (12 points) Evaluate the integral

$$\mathrm{pv} \int_{-\infty}^{\infty} \frac{x}{(x-\mathrm{i})(x+2\mathrm{i})(x-3\mathrm{i})(x+4\mathrm{i})} \,\mathrm{d}x,$$

using the calculus of residues. Give complete arguments.

Question 3 (20 points)

Represent the function

$$f(z) = \frac{z}{z^2 - 1},$$

(a) (8 points) as a Taylor series around 0 and find its radius of convergence;

(b) (7 points) as a Laurent series in the domain $|z| > 1$.

(c) (5 points) Determine the singularities of the function

$$g(z) = \frac{\sin z}{z},$$

and their type (removable, pole, essential; if pole, give the order). Justify your answers.

Question 4 (15 points)

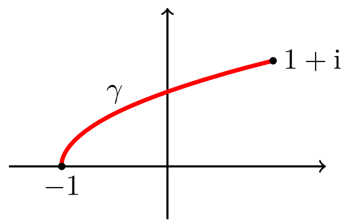
Consider the polynomial $P(z) = z^4 + \varepsilon(z-1)$ where $\varepsilon > 0$. Show that if $\varepsilon < \dfrac{r^4}{1+r}$ then the polynomial P has four zeros inside the circle $|z| = r$.

Question 5 (15 points)

Compute the following integrals along the path γ shown below that lies in the upper half-plane, starts at -1 and ends at $1 + \mathrm{i}$. Give complete arguments.

(a) (6 points) $\displaystyle\int_\gamma \frac{1}{z^2}\,\mathrm{d}z$.

(b) (9 points) $\displaystyle\int_\gamma \frac{1}{z}\,\mathrm{d}z$.



Question 6 (10 points)

Answer only one of the following two questions:

Question A. Consider the Möbius transformation $f(z) = \dfrac{1+\mathrm{i}}{-\mathrm{i}z+1}$ on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. After computing $f(0)$, $f(\pm 1)$, and $f(\pm\mathrm{i})$, and expressing them in standard form (real plus imaginary part), determine the image of the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ under f.

Question B. Prove that if $f(z)$ is entire and agrees with a polynomial $\sum_{j=0}^{n} a_j x^j$ for $z = x$ on a segment of the real axis, then $f(z) = \sum_{j=0}^{n} a_j z^j$ everywhere.

Formulas

The Cauchy-Riemann equations for a function $f = u + \mathrm{i}v$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The principal value of the logarithm is

$$\operatorname{Log} z = \operatorname{Log}|z| + \mathrm{i}\operatorname{Arg} z.$$

The generalized Cauchy integral formula is

$$f^{(n)}(z_0) = \frac{n!}{2\pi\mathrm{i}} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

The residue of a function f at a pole z_0 of order m is given by

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z-z_0)^m f(z)\right].$$